

# **$j$ -MULTIPLICITY AND DEPTH OF ASSOCIATED GRADED MODULES**

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**ABSTRACT.** Let  $R$  be a Noetherian local ring. We define the minimal  $j$ -multiplicity and almost minimal  $j$ -multiplicity of an arbitrary  $R$ -ideal on any finite  $R$ -module. For any ideal  $I$  with minimal  $j$ -multiplicity or almost minimal  $j$ -multiplicity on a Cohen-Macaulay module  $M$ , we prove that under some residual assumptions, the associated graded module  $\mathrm{gr}_I(M)$  is Cohen-Macaulay or almost Cohen-Macaulay, respectively. Our work generalizes the results for minimal multiplicity and almost minimal multiplicity obtained by Sally, Rossi, Valla, Wang, Huckaba, Elias, Corso, Polini, and VazPinto.

## 1. INTRODUCTION

In this paper we investigate the behavior of the depth of the associated graded ring  $\mathrm{gr}_I(R)$  of an ideal  $I$  in a Noetherian local ring  $(R, \mathfrak{m})$  in terms of conditions on the  $j$ -multiplicity of  $I$ . The associated graded ring of  $I$  is an algebraic construction whose projective scheme represents the exceptional fiber of the blowup of a variety along a subvariety. Its arithmetical properties, like its depth, provide useful information, for instance, on the cohomology groups of the blowup. For an  $\mathfrak{m}$ -primary ideal  $I$ , the interplay between the Hilbert polynomial of  $I$ , and more precisely its Hilbert coefficients, and the depth of the associated graded ring has been widely investigated. This line of study has its roots in the pioneering work of Sally. The idea is that extremal values of the Hilbert coefficients, most notably of the multiplicity of  $I$ , yield high depth of the associated graded ring and, conversely, good depth properties encode information about all the Hilbert coefficients, such as their positivity. The problem arises when one considers ideals which are not  $\mathfrak{m}$ -primary, because their Hilbert function is not defined, thus there is no numerical information on Hilbert coefficients available to study the Cohen-Macaulayness of  $\mathrm{gr}_I(R)$ . To remedy the lack of this tool, in this paper we propose to use the notion of  $j$ -multiplicity. The  $j$ -multiplicity was developed as a generalization of the Hilbert-Samuel multiplicity to arbitrary ideals. It was first introduced by Achilles and Manaresi in 1993 and, since then, it has been frequently used by both algebraists and geometers as an invariant to deal with improper intersections and varieties with non isolated singularities [2].

In this introduction we will only discuss the case of associated graded rings, although in the rest of the paper we will treat associated graded modules.

Let  $I$  be an  $\mathfrak{m}$ -primary ideal. The Hilbert-Samuel function of  $I$  is the numerical function  $H_I(n)$  that measures the growth of the length  $\lambda(R/I^n)$  of the powers of  $I$  for all  $n \geq 1$ . For  $n$  sufficiently

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large, the Hilbert-Samuel function is a polynomial function in  $n$  of degree  $d$ , the dimension of  $R$ . This is the Hilbert-Samuel polynomial of  $I$ , whose coefficients  $e_i(I)$ , dubbed the Hilbert coefficients of  $I$ , are uniquely determined by  $I$ . It is well known that the normalized leading coefficient  $e_0(I)$ , the multiplicity of  $I$ , detects integral dependence of  $\mathfrak{m}$ -primary ideals. The integral closure of  $I$ , for instance, can be characterized as the largest ideal containing  $I$  with the same multiplicity  $e_0$ , when the ring is equidimensional and universally catenary. Flenner and Manaresi were the first to use the  $j$ -multiplicity to generalize this fundamental theorem of Rees to arbitrary ideals [5].

In 1967 Abhyankar proved that the multiplicity  $e(R) = e_0(\mathfrak{m})$  of a  $d$ -dimensional Cohen-Macaulay local ring is bounded below by  $\mu(\mathfrak{m}) - d + 1$ , where  $\mu(\mathfrak{m})$  is the embedding dimension of  $R$  [1]. Rings for which  $e_0 = \mu(\mathfrak{m}) - d + 1$  have since then been called rings of minimal multiplicity. In the case of minimal multiplicity, Sally had shown in [21] that the associated graded ring of  $\mathfrak{m}$  is always Cohen-Macaulay. Even if the multiplicity is almost minimal, the associated graded ring is Gorenstein provided the ambient ring is Gorenstein [22]. Unfortunately, for arbitrary Cohen-Macaulay rings of almost minimal multiplicity the Cohen-Macaulayness of  $\text{gr}_{\mathfrak{m}}(R)$  fails to hold, the exceptions being Cohen-Macaulay local rings of maximal type [23]. Based on this result, Sally conjectured that if the multiplicity of  $R$  is almost minimal then the depth of the associated graded ring is almost maximal, i.e., it is at least  $d - 1$ . This conjecture was proved almost twenty years later by Rossi and Valla [18], and independently by Wang [28]. In recent years there have been many generalizations of these results to  $\mathfrak{m}$ -primary ideals and modules with  $I$ -adic filtrations, where  $I$  is an ideal of definition, a condition that is required to define the Hilbert-Samuel multiplicity (see for example [19], [9], [3], [4], [17], [14], [20]).

The kind of generalization we accomplish in this article is much harder. We investigate the depth of the associated graded ring of an arbitrary ideal using the  $j$ -multiplicity introduced by Achilles and Manaresi [2] and further studied in [7], [5], [6] and [13]. Let  $(R, \mathfrak{m})$  be a Noetherian local ring of dimension  $d > 0$  and  $I$  an  $R$ -ideal. One can assign generalized Hilbert coefficients  $j_i(F)$  to every ideal filtration  $F$  whose Rees algebra is finite over  $R[It] = \bigoplus_{i \geq 0} I^i t^i$  in the following way: let  $A$  be the associated graded ring of  $F$ , and denote by  $\Gamma_{\mathfrak{m}}(A) = H_{\mathfrak{m}}^0(A)$  the submodule of elements supported on  $\mathfrak{m}$ . Since  $\Gamma_{\mathfrak{m}}(A)$  is annihilated by a large power of  $\mathfrak{m}$ , it is a finite graded module over  $\text{gr}_I(R) \otimes R/\mathfrak{m}^k$  for some  $k$ , hence its Hilbert polynomial is well defined:

$$P(n) = \sum_{i=0}^{d-1} (-1)^i j_i(F) \binom{n+d-i-1}{d-i-1}.$$

We call  $P(n)$  the *generalized Hilbert polynomial* of  $F$ . The generalized Hilbert coefficients of the filtration  $\{I^n\}_{n \in \mathbb{N}}$  will simply be denoted by  $j_i(I)$ . Notice that  $j_0(I)$  coincides with the  $j$ -multiplicity defined by Achilles and Manaresi in [2]. Furthermore in the  $\mathfrak{m}$ -primary case  $j_i(I) = e_i(I)$ , so our definition recovers the standard one.

In Section 2, we prove a lower bound for the  $j$ -multiplicity of any ideal  $I$ . The definition of ideals with minimal  $j$ -multiplicity is thus immediate. In Section 3, under certain Artin-Nagata condition, we prove that for any ideal with minimal  $j$ -multiplicity, the associated graded ring is

Cohen-Macaulay. Furthermore if the ambient ring is Gorenstein, then the associated graded ring is Gorenstein as well, which generalizes completely Sally's results. Finally, in Section 4, we deal with ideals with almost minimal  $j$ -multiplicity. We prove that, under the same residual assumptions, the associated graded ring is almost Cohen-Macaulay. The technical novelty is a powerful combination of the methods used in the  $\mathfrak{m}$ -primary case with tools proper to residual intersection theory. This result can be viewed as a positive answer to Sally's conjecture for arbitrary ideals.

## 2. MINIMAL $j$ -MULTIPLICITY

In this section we first prove a lower bound for the  $j$ -multiplicity; this bound leads to a notion of minimal  $j$ -multiplicity.

We start by fixing the notation that will be used throughout the paper. We first recall the definition of  $j$ -multiplicity according to [2] and [13].

Let  $(R, \mathfrak{m})$  be a Noetherian local ring,  $I$  an arbitrary  $R$ -ideal, and  $M$  a finite  $R$ -module of dimension  $d$ . The  $I$ -adic filtration of  $M$  is a collection of submodules  $\{I^j M\}_{j \geq 0}$ . Let  $G = \text{gr}_I(R) = \bigoplus_{j=0}^{\infty} I^j / I^{j+1}$  be the associated graded ring of  $I$  and  $T = \text{gr}_I(M) = \bigoplus_{j=0}^{\infty} I^j M / I^{j+1} M$  the associated graded module of the filtration  $\{I^j M\}_{j \geq 0}$ . Notice that  $T$  is a finite graded module over the graded ring  $G$ . In general the homogeneous components of  $T$  may not have finite length, thus we consider the  $T$ -submodule of elements supported on  $\mathfrak{m}$ ,  $W = \Gamma_{\mathfrak{m}}(T) = 0 :_T \mathfrak{m}^{\infty} = \bigoplus_{j=0}^{\infty} \Gamma_{\mathfrak{m}}(I^j M / I^{j+1} M)$ . Since  $W$  is annihilated by a large power of  $\mathfrak{m}$ , it is a finite graded module over  $\text{gr}_I(R) \otimes R/\mathfrak{m}^k$  for some  $k$ , hence its Hilbert polynomial  $P(n)$  is well defined. Notice that  $\dim_G W \leq \dim_G T = d$ , thus  $P(n)$  has degree at most  $d - 1$ . The  $j$ -multiplicity of  $I$  on  $M$  is the normalized coefficient of  $P(n)$  in degree  $d - 1$ ,

$$j(I, M) = (d - 1)! \lim_{t \rightarrow \infty} \frac{\lambda(\Gamma_{\mathfrak{m}}(I^t M / I^{t+1} M))}{t^{d-1}}.$$

Recall that the Krull dimension of the special fiber module  $T/\mathfrak{m}T$  is called the *analytic spread* of  $I$  on  $M$  and is denoted by  $\ell(I, M)$ . In general,  $\dim_G W \leq \ell(I, M) \leq d$  and equalities hold if  $\ell(I, M) = d$ . Therefore  $j(I, M) \neq 0$  if and only if  $\ell(I, M) = d$  [13, 2.1].

If  $M/IM$  has finite length, the ideal  $I$  is said to be an *ideal of definition* for  $M$ . In this case each homogeneous component of  $T$  has finite length, thus  $W = T$  and the  $j$ -multiplicity coincides with the usual multiplicity.

An element  $x \in I$  is said to be a *superficial element* for  $I$  on  $M$  if there exists a non-negative integer  $c$  such that

$$(I^{t+1} M :_M x) \cap I^c M = I^t M.$$

A sequence of elements  $x_1, \dots, x_s$  in  $I$  is a *superficial sequence* for  $I$  on  $M$  if  $x_i$  is superficial for  $I$  on  $M/(x_1, \dots, x_{i-1})M$  for  $i = 1, \dots, s$ . This notion, originally introduced by Zariski and Samuel, plays a significant role in the study of Hilbert functions because it allows to reduce the problems to lower dimensional ones. Notice that if  $M$  has positive depth then every superficial element is regular on  $M$ . If  $d = \dim M \geq 1$  then a superficial element has always order one, i.e.,  $x \in I/I^2$ . Thus, in this case, the definition of superficial elements coincides with the definition of homogeneous filter regular

elements used in the study of  $j$ -multiplicity (see [2], [29], and [13] for instance). More precisely, an element  $x$  is superficial for  $I$  on  $M$  if and only if  $x^*$ , the image of  $x$  in  $I/I^2$ , is *filter-regular* for  $G_+$  on  $T$ .

For an ideal  $J \subseteq I$ , one says that  $J$  is a *reduction* of  $I$  on  $M$  if  $JI^t M = I^{t+1} M$  for some non negative integer  $t$ . A *minimal* reduction is a reduction which is minimal with respect to inclusion. Minimal reductions always exist and, if  $R$  has infinite residue field, the minimal number of generators of any minimal reduction  $J$  of  $I$  on  $M$  equals the analytic spread  $\ell(I, M)$ . Furthermore, a minimal generating set of  $J$  can be chosen to be a superficial sequence for  $I$  on  $M$  [24, 3.1]. The least integer  $t$  with  $JI^t M = I^{t+1} M$  is called the *reduction number* of  $I$  on  $M$  with respect to  $J$  and denoted by  $r_J(I, M)$ . One then defines the *reduction number*  $r(I, M)$  of  $I$  on  $M$  to be the least  $r_J(I, M)$ , where  $J$  varies over all minimal reductions of  $I$  on  $M$ .

Let  $I = (a_1, \dots, a_n)$  and write  $x_i = \sum_{j=1}^n \lambda_{ij} a_j$  for  $1 \leq i \leq s$  and  $(\lambda_{ij}) \in R^{sn}$ . The elements  $x_1, \dots, x_s$  form a *sequence of general elements* in  $I$  (equivalently  $x_1, \dots, x_s$  are *general* in  $I$ ) if there exists a dense open subset  $U$  of  $k^{sn}$  such that the image  $(\overline{\lambda_{ij}}) \in U$ . In this case we call the  $(\lambda_{ij})$  general elements in  $R^{sn}$ . When  $s = 1$  we say that  $x = x_1$  is *general* in  $I$ . Observe that the notion of general elements of  $I$  is more restrictive than the one of *sequentially general* elements. The latter means that for every  $i$  with  $1 \leq i \leq s$  and every fixed  $x_1, \dots, x_{i-1}$  the element  $x_i$  is general in  $I$ .

The notion of general elements is a fundamental tool for our study as they are always a superficial sequence for  $I$  on  $M$  [29, 2.5]; they generate a minimal reduction whose reduction number  $r_J(I, M)$  coincides with the reduction number  $r(I, M)$  of  $I$  on  $M$  if  $s = \ell(I, M)$  (see [25, 2.2] and [11, 8.6.6]); and they form a *super-reduction* in the sense of [2] whenever  $s = \ell(I, M) = d = \dim_R M$  (see [29, 2.5]). Furthermore, one can compute the  $j$ -multiplicity using general elements and obtain a lower bound from it as the next proposition shows.

**Proposition 2.1.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring with infinite residue field  $k$ . Let  $M$  be a finite  $R$ -module and  $I$  an  $R$ -ideal with analytic spread  $\ell(I, M) = d = \dim_R M$ . Then for general elements  $x_1, \dots, x_d$  in  $I$ , we have*

(a) *the  $j$ -multiplicity of  $I$  on  $M$  is*

$$j(I, M) = e(I, M / ((x_1, \dots, x_{d-1})M :_M I^\infty)) = \lambda(M / ((x_1, \dots, x_{d-1})M :_M I^\infty + x_d M));$$

(b)  *$j(I, M) \geq \lambda(\overline{IM} / I^2 \overline{M})$  where  $\overline{M} = M / ((x_1, \dots, x_{d-1})M :_M I^\infty)$ .*

**Proof.** By [29, 2.5], there exist general elements  $x_1, \dots, x_d$  in  $I$  which form a super-reduction in the sense of [2]. Now proceed as in the proof of [2, 3.8] or [13, 3.6] to obtain the desired formula of part (a). For part (b), as  $\overline{M}$  is a one-dimensional Cohen-Macaulay module and  $I$  is an ideal of definition for  $\overline{M}$ , we have  $j(I, M) = e(I, \overline{M}) = \lambda(\overline{IM} / I^2 \overline{M}) + \lambda(I^2 \overline{M} / x_d \overline{IM}) \geq \lambda(\overline{IM} / I^2 \overline{M})$ , where the second equality follows from [20, Corollary 2.1].  $\square$

Once we show that  $\lambda(\overline{IM} / I^2 \overline{M})$  and  $\lambda(I^2 \overline{M} / x_d \overline{IM})$  do not depend on the choice of the general elements  $x_1, \dots, x_d$  in  $I$ , we will obtain the desired lower bound for the  $j$ -multiplicity.

**Lemma 2.2.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring with infinite residue field  $k$ . Let  $R' = R[z_1, \dots, z_t]$  be a polynomial ring over  $R$ . Let  $M' \supseteq N'$  be two finite  $R'$ -modules with  $\lambda_{R'/\mathfrak{m}_{R'}}(M'_{\mathfrak{m}_{R'}}/N'_{\mathfrak{m}_{R'}}) = s$ . If  $(\Lambda) = (\lambda_1, \dots, \lambda_t)$  is a vector in  $k^t$ , write  $(\overline{\Lambda})$  for its image in  $k^t$  and  $\pi(\cdot)$  for the evaluation map sending  $z_i$  to  $\lambda_i$ . Then there exists a dense open subset  $U$  of  $k^t$  such that  $\lambda_R(\pi(M')/\pi(N')) \leq s$  whenever  $(\overline{\Lambda}) \in U$ .*

**Proof.** If  $s = \infty$  then we are done. So we may assume  $s < \infty$  and let  $M' = M'_0 \supset M'_1 \supset \dots \supset M'_s = N'$  be a filtration such that  $(M'_{l-1}/M'_l)_{\mathfrak{m}_{R'}} \cong k(z_1, \dots, z_t)$  for  $1 \leq l \leq s$ . We use induction on  $s$  to prove the lemma. When  $s = 0$ , i.e.,  $\lambda_{R'/\mathfrak{m}_{R'}}(M'_{\mathfrak{m}_{R'}}/N'_{\mathfrak{m}_{R'}}) = 0$ , there exists a polynomial  $f \in R' \setminus \mathfrak{m}_{R'}$  such that  $fM' \subseteq N'$ . Let  $\overline{f}$  be the image of  $f$  in  $k[z_1, \dots, z_t]$  and notice that  $\overline{f} \neq 0$ . Thus  $U = D(\overline{f})$  is a dense open subset of  $k^t$ . If  $(\overline{\Lambda}) \in U$  then  $f(\Lambda)$  is a unit in  $R$ . Thus  $f(\Lambda)\pi(M') = \pi(fM') \subseteq \pi(N')$  implies  $\pi(M') \subseteq \pi(N')$ . Now assume the lemma holds for  $s - 1$ , i.e., there exists a dense open subset  $U_1 \subseteq k^t$  such that  $\lambda_R(\pi(M')/\pi(M'_{s-1})) \leq s - 1$  whenever  $(\overline{\Lambda}) \in U_1$ . Since  $(M'_{s-1}/M'_s)_{\mathfrak{m}_{R'}} \cong k(z_1, \dots, z_t)$ , there exists  $b' \in M'_{s-1}$  and a polynomial  $f \in R' \setminus \mathfrak{m}_{R'}$  so that  $fM'_{s-1} \subseteq R'b' + M'_s$  and  $fmb' \in M'_s$ . Also notice that the image  $\overline{f}$  of  $f$  in  $k[z_1, \dots, z_t]$  is not zero and  $U_2 = D(\overline{f})$  is a dense open subset of  $k^t$ . Let  $U = U_1 \cap U_2$ . Whenever  $(\overline{\Lambda}) \in U$ ,  $f(\Lambda)$  is a unit in  $R$ . Thus  $\pi(M'_{s-1}) \subseteq R\pi(b') + \pi(M'_s)$  and  $\mathfrak{m}\pi(b') \in \pi(M'_s)$ . Therefore  $\lambda_R(\pi(M'_{s-1})/\pi(M'_s)) \leq 1$  and thus we obtain  $\lambda_R(\pi(M')/\pi(N')) = \lambda_R(\pi(M')/\pi(M'_{s-1})) + \lambda_R(\pi(M'_{s-1})/\pi(M'_s)) \leq s$  for every  $(\overline{\Lambda}) \in U$ .  $\square$

**Lemma 2.3.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring with infinite residue field  $k$ . Let  $M$  be a finite  $R$ -module and  $I$  an  $R$ -ideal with analytic spread  $\ell(I, M) = d = \dim_R M$ . For  $x_1, \dots, x_d$  general elements in  $I$ , write  $\overline{M} = M/((x_1, \dots, x_{d-1})M :_M I^\infty)$ , then the lengths  $\lambda(I\overline{M}/I^2\overline{M})$  and  $\lambda(I^2\overline{M}/x_d I\overline{M})$  are independent of  $x_1, \dots, x_d$ .*

**Proof.** Let  $a_1, \dots, a_n$  be a set of generators of  $I$  and  $Z = (z_{ij})$  be  $d \times n$  variables. Write  $R' = R[Z]$ ,  $x'_i = \sum_{j=1}^n z_{ij}a_j$  for  $1 \leq i \leq d$ , and  $M' = M \otimes_R R'$ . Let  $\overline{M'} = M'/((x'_1, \dots, x'_{d-1})M' :_{M'} IR'^\infty)$ , by Proposition 2.1(a) and the proof of Proposition 2.1(b) (see also [2, 3.8], [13, 3.6] and [20, Corollary 2.1]) we have

$$j(I, M) = j(IR'_{\mathfrak{m}_{R'}}, M'_{\mathfrak{m}_{R'}}) = \lambda(I\overline{M'}_{\mathfrak{m}_{R'}}/I^2\overline{M'}_{\mathfrak{m}_{R'}}) + \lambda(I^2\overline{M'}_{\mathfrak{m}_{R'}}/x'_d I\overline{M'}_{\mathfrak{m}_{R'}}).$$

For general elements  $(\Lambda) = (\lambda_{ij}) \in R'^{dn}$ , write  $\pi(\cdot)$  for the evaluation map sending  $z_{ij}$  to  $\lambda_{ij}$ . Observe  $\pi(IM') = IM$ ,  $\pi(I^2M') = I^2M$ ,  $\pi(x'_d IM') = x_d IM$ , and clearly

$$\pi((x'_1, \dots, x'_{d-1})M' :_{M'} IR'^\infty) \subseteq (x_1, \dots, x_{d-1})M :_M I^\infty.$$

Putting this together with Lemma 2.2, we obtain

$$\begin{aligned} \lambda(I\overline{M'}_{\mathfrak{m}_{R'}}/I^2\overline{M'}_{\mathfrak{m}_{R'}}) &= \lambda(IM'_{\mathfrak{m}_{R'}}/[(x'_1, \dots, x'_{d-1})M'_{\mathfrak{m}_{R'}} :_{IM'_{\mathfrak{m}_{R'}}} (IR'_{\mathfrak{m}_{R'}})^\infty + I^2M'_{\mathfrak{m}_{R'}}]) \\ &\geq \lambda(IM/[\pi((x'_1, \dots, x'_{d-1})M' :_{IM'} IR'^\infty) + I^2M]) \\ &\geq \lambda(IM/[(x_1, \dots, x_{d-1})M :_{IM} I^\infty + I^2M]) = \lambda(I\overline{M}/I^2\overline{M}). \end{aligned}$$

In the same way we have,

$$\begin{aligned}\lambda(I^2 \overline{M}'_{\mathfrak{m}R'} / x'_d \overline{I} \overline{M}'_{\mathfrak{m}R'}) &= \lambda(I^2 M'_{\mathfrak{m}R'} / [(x'_1, \dots, x'_{d-1}) M'_{\mathfrak{m}R'} :_{I^2 M'_{\mathfrak{m}R'}} (IR'_{\mathfrak{m}R'})^\infty + x'_d IM'_{\mathfrak{m}R'}]) \\ &\geq \lambda(I^2 M'_{\mathfrak{m}R'} / [(x_1, \dots, x_{d-1}) M :_{I^2 M} I^\infty + x_d IM]) = \lambda(I^2 \overline{M} / x_d \overline{I} \overline{M}).\end{aligned}$$

By Proposition 2.1(a) and the proof of Proposition 2.1(b), the  $j$ -multiplicity is given by the sum of  $\lambda(\overline{I} \overline{M} / I^2 \overline{M})$  and  $\lambda(I^2 \overline{M} / x_d \overline{I} \overline{M})$ , thus

$$\begin{aligned}j(I, M) &= \lambda(\overline{I} \overline{M} / I^2 \overline{M}) + \lambda(I^2 \overline{M} / x_d \overline{I} \overline{M}) \\ &\leq \lambda(\overline{I} \overline{M}'_{\mathfrak{m}R'} / I^2 \overline{M}'_{\mathfrak{m}R'}) + \lambda(I^2 \overline{M}'_{\mathfrak{m}R'} / x'_d \overline{I} \overline{M}'_{\mathfrak{m}R'}) \\ &= j(I, M).\end{aligned}$$

In turn this forces the equalities

$$\begin{aligned}\lambda(\overline{I} \overline{M} / I^2 \overline{M}) &= \lambda(\overline{I} \overline{M}'_{\mathfrak{m}R'} / I^2 \overline{M}'_{\mathfrak{m}R'}), \\ \lambda(I^2 \overline{M} / x_d \overline{I} \overline{M}) &= \lambda(I^2 \overline{M}'_{\mathfrak{m}R'} / x'_d \overline{I} \overline{M}'_{\mathfrak{m}R'}),\end{aligned}$$

and therefore shows the independence of these lengths from the general elements  $x_1, \dots, x_d$ .  $\square$

Because of Proposition 2.1 and Lemma 2.3, we can now give the definition of *minimal  $j$ -multiplicity of  $I$  on  $M$*  which is the analogue of minimal multiplicity [20].

**Definition 2.4.** Let  $M$  be a finite module of dimension  $d$  over a Noetherian local ring  $R$  and  $I$  an  $R$ -ideal with analytic spread  $\ell(I, M) = d$ . We say that  $I$  has *minimal  $j$ -multiplicity on  $M$*  if  $j(I, M) = \lambda(\overline{I} \overline{M} / I^2 \overline{M})$ , where  $\overline{M} = M / ((x_1, \dots, x_{d-1}) M :_M I^\infty)$  and  $x_1, \dots, x_{d-1}$  are general in  $I$ .

Notice that  $I$  has minimal  $j$ -multiplicity on  $M$  if and only if  $\lambda(I^2 \overline{M} / x_d \overline{I} \overline{M})$  is zero, or equivalently, if and only if  $x_d$  generates a reduction of  $I$  on  $\overline{M}$  with reduction number one. The next lemma shows that for an ideal the assumption of having minimal  $j$ -multiplicity on  $M$  is quite strict. Indeed, if  $I$  has minimal  $j$ -multiplicity on  $M$  then the Hilbert function of  $I$  on  $\overline{M}$  is rigid, i.e., the value of the multiplicity determines the Hilbert function. We remark that results of this kind are really surprising since the multiplicity is just one of the Hilbert coefficients and, in turn, the Hilbert coefficients give only partial information on the Hilbert polynomial which gives only asymptotic information on the Hilbert function.

**Corollary 2.5.** Let  $R, I, M$  and  $\overline{M}$  be as in Definition 2.4, then the  $j$ -multiplicity of  $I$  on  $M$  and on  $I^t M$  coincides for every  $t \geq 0$ , i.e.,  $j(I, M) = j(I, I^t M)$  for every  $t \geq 0$ . Furthermore, if  $I$  has minimal  $j$ -multiplicity on  $M$  then  $j(I, M) = \lambda(I^t \overline{M} / I^{t+1} \overline{M})$  for every  $t \geq 1$ .

**Proof.** Observe that  $\ell(I, M) = \dim T / \mathfrak{m}T = \dim T_{\geq t} / \mathfrak{m}T_{\geq t} = \ell(I, I^t M)$  and if  $x_1, \dots, x_d$  are general elements of  $I$  on  $M$ , they are also general on  $I^t M$  for all  $t \geq 0$ . The first assertion follows because

$(x_1, \dots, x_{d-1})M :_{IM} I^\infty = (x_1, \dots, x_{d-1})IM :_{IM} I^\infty$ , thus

$$\begin{aligned} j(I, M) &= \lambda(IM / ((x_1, \dots, x_{d-1})M :_{IM} I^\infty + x_d IM)) \\ &= \lambda(IM / ((x_1, \dots, x_{d-1})IM :_{IM} I^\infty + x_d IM)) \\ &= j(I, IM). \end{aligned}$$

Now continue the above process, we will obtain  $j(I, M) = j(I, I^t M)$  for every  $t \geq 0$ .

For the second assertion assume that  $I$  has minimal  $j$ -multiplicity on  $M$ , or equivalently,  $I^2 \overline{M} = x_d \overline{IM}$ . This gives  $I^{t+2} \overline{M} = x_d I^{t+1} \overline{M}$  for every  $t \geq 0$ . In turn, this implies that  $x_d$  generates a reduction of  $I$  on  $I^t \overline{M}$  with reduction number one, thus  $I$  has minimal  $j$ -multiplicity on  $I^t M$  and therefore  $j(I, I^t M) = \lambda(I^t \overline{M} / I^{t+2} \overline{M}) = \lambda(I^{t+1} \overline{M} / I^{t+2} \overline{M})$ . Hence  $j(I, M) = j(I, I^t M) = \lambda(I^{t+1} \overline{M} / I^{t+2} \overline{M})$  for every  $t \geq 0$ .  $\square$

### 3. COHEN-MACAULAYNESS OF THE ASSOCIATED GRADED MODULE

In this section we show that the associated graded module of any filtration with minimal  $j$ -multiplicity is Cohen-Macaulay, if the ideal has some residual properties. We start by describing the residual assumptions that are needed to prove the main theorem.

Let  $M$  be a finite faithful module of dimension  $d$  over a Noetherian local ring  $R$ . Let  $I$  be an  $R$ -ideal. The ideal  $I$  is said to satisfy the *condition  $G_s$*  on  $M$  if for every  $\mathfrak{p} \in \text{Supp}_R(M/IM)$  with  $\dim R_{\mathfrak{p}} = t < s$  the ideal  $I$  is generated by  $t$  element on  $M_{\mathfrak{p}}$ , i.e.,  $IM_{\mathfrak{p}} = (x_1, \dots, x_t)M_{\mathfrak{p}}$  for some  $x_1, \dots, x_t$  in  $I$ . Write  $H = (x_1, \dots, x_t)M :_M I$ . If  $IM_{\mathfrak{p}} = (x_1, \dots, x_t)M_{\mathfrak{p}}$  for every  $\mathfrak{p} \in \text{Spec}(R)$  with  $\dim R_{\mathfrak{p}} \leq t - 1$ , then  $H$  is said to be a  *$t$ -residual intersection* of  $I$  on  $M$ . Now let  $H$  be a  $t$ -residual intersection of  $I$  on  $M$ . If in addition  $IM_{\mathfrak{p}} = (x_1, \dots, x_t)M_{\mathfrak{p}}$  for every  $\mathfrak{p} \in \text{Supp}_R(M/IM)$  with  $\dim R_{\mathfrak{p}} \leq t$ , then  $H$  is said to be a *geometric  $t$ -residual intersection* of  $I$  on  $M$ . If  $M$  is not faithful, then we say that  $I$  satisfies the *condition  $G_s$*  on  $M$  if  $\overline{I\overline{R}}$  satisfies the condition  $G_s$  on  $M$ , where  $\overline{R} = R/\text{Ann } M$ . We say  $H$  is a  *$t$ -residual intersection* or *geometric  $t$ -residual intersection* of  $I$  on  $M$  if  $H$  is a  $t$ -residual intersection or geometric  $t$ -residual intersection of  $\overline{I\overline{R}}$  on  $M$  respectively.

The next two lemmas contain basic facts about residual intersections. The ideas are already presented in [26, 1.6 and 1.7]. The first lemma says that the condition  $G_s$  gives rise to residual intersections.

**Lemma 3.1.** *Let  $R$  be a catenary and equidimensional Noetherian local ring with infinite residue field. Let  $M$  be a finite  $R$ -module and  $I$  an  $R$ -ideal satisfying condition  $G_s$  on  $M$ . For general elements  $x_1, \dots, x_s$  of  $I$  on  $M$ , write  $H_i = (x_1, \dots, x_i)M :_M I$  for  $0 \leq i \leq s$ , then:*

- (a)  $H_i$  is an  $i$ -residual intersection of  $I$  on  $M$  for  $0 \leq i \leq s$ .
- (b)  $H_i$  is a geometric  $i$ -residual intersection of  $I$  on  $M$  for  $0 \leq i \leq s - 1$ .

**Proof.** Let  $a_1, \dots, a_n$  be a set of generators of  $I$  on  $M$ , i.e.,  $IM = (a_1, \dots, a_n)M$ , and  $Z = (z_{ij})$  be  $s \times n$  variables. Write  $R' = R[Z]_{\mathfrak{m}R[Z]}$  and  $M' = M \otimes_R R'$ , where  $\mathfrak{m}$  is the maximal ideal of  $R$ . For  $1 \leq i \leq s$ , let  $x'_i = \sum_{j=1}^n z_{ij} a_j$ . We first claim that

( $\star$ ) if  $\mathfrak{p} \in \text{Supp}(M/IM)$  such that  $IM_{\mathfrak{p}} = (y_1, \dots, y_l)M_{\mathfrak{p}}$  for some  $y_1, \dots, y_l \in I$  where  $1 \leq l \leq s$ , then  $IM'_{\mathfrak{p}} = (x'_1, \dots, x'_l)M'_{\mathfrak{p}}$ .

To prove it, let  $X_1, \dots, X_n$  be variables over  $R'_{\mathfrak{p}}/\mathfrak{p}R'_{\mathfrak{p}}$ . Set  $A' = M'_{\mathfrak{p}}/\mathfrak{p}M'_{\mathfrak{p}}[X_1, \dots, X_n]$ ,  $T'_{\mathfrak{p}} = \text{gr}_{IR'_{\mathfrak{p}}}(M'_{\mathfrak{p}})$ , and  $F' = T'_{\mathfrak{p}}/\mathfrak{p}T'_{\mathfrak{p}}$ . Define the map  $\varphi : A' \rightarrow F'$  by sending  $X_i$  to  $a_i + \mathfrak{p}IR'_{\mathfrak{p}}$ . For  $1 \leq i \leq l$ , write  $y_i = \sum_{j=1}^n \lambda_{ij}a_j$ , where  $(\lambda_{ij}) \in R^{ln}$ . Denote with  $\bar{\phantom{x}}$  the images of elements of  $R$  in  $R'_{\mathfrak{p}}/\mathfrak{p}R'_{\mathfrak{p}}$ . Observe that the preimages of the  $y_i$ 's generate a vector space in  $[A']_1$  of dimension  $n$ , i.e.,  $\dim \text{Span}\{\{b_i = \sum_{j=1}^n \bar{\lambda}_{ij}X_j\}_{1 \leq i \leq l}, [\ker(\varphi)]_1\} = n$ . To show that  $IM'_{\mathfrak{p}} = (x'_1, \dots, x'_l)M'_{\mathfrak{p}}$  it will be enough to show that the set  $\{\{b'_i = \sum_{j=1}^n z_{ij}X_j\}_{1 \leq i \leq l}, [\ker(\varphi)]_1\}$  spans also a  $n$ -dimensional vector space in  $[A']_1$ , i.e.,  $\dim \text{Span}\{\{b'_i = \sum_{j=1}^n z_{ij}X_j\}_{1 \leq i \leq l}, [\ker(\varphi)]_1\} = n$ . Indeed, define the map

$$\phi : A = M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}[X_1, \dots, X_n] \rightarrow F = \text{gr}_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})/(\mathfrak{p}\text{gr}_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}))$$

by sending  $X_i$  to  $a_i + \mathfrak{p}IR_{\mathfrak{p}}$ . Since the extension from  $R$  to  $R'$  is flat,  $\ker(\varphi) = \ker(\phi) \otimes_R R'$ . Thus we can choose basis  $b_i = \sum_{j=1}^n \bar{\lambda}_{ij}X_j$  in  $[\ker(\phi)]_1$  where  $\lambda_{ij} \in R$  and  $l+1 \leq i \leq t$  such that

$$\dim \text{Span}(b_1, \dots, b_{l+t}) = n.$$

This forces  $\dim \text{Span}(b'_1, \dots, b'_l, b_{l+1}, \dots, b_{l+t}) = n$  as well. If not, set

$$\alpha = \dim \text{Span}(b'_1, \dots, b'_l, b_{l+1}, \dots, b_{l+t}) < n.$$

Observe that each  $b'_i$  is a linear combination of  $X_j$  with coefficients polynomials in the variable  $z_{ij}$  of degree at most one, and let  $X$  be the  $(l+t) \times n$  matrix obtained from these coefficients. Because  $\alpha < n$  then all the  $n \times n$  minors of  $X$  vanish. When we specialize  $z_{ij}$  to  $\lambda_{ij}$  for  $1 \leq i \leq l$ , then all the  $n \times n$  minors of the  $(l+t) \times n$  matrix obtained from the coefficients of  $b_1, \dots, b_l, b_{l+1}, \dots, b_{l+t}$  as linear combinations of  $X_j$  vanish as well, which contradicts the fact that  $\dim \text{Span}(b_1, \dots, b_{l+t}) = n$ .

Write  $H'_i = (x'_1, \dots, x'_i)M' :_{M'} IR'$  for  $0 \leq i \leq s$ . We claim that:

(a')  $H'_i$  is an  $i$ -residual intersection of  $IR'$  on  $M'$  for  $0 \leq i \leq s$ .

(b')  $H'_i$  is a geometric  $i$ -residual intersection of  $IR'$  on  $M'$  for  $0 \leq i \leq s-1$ .

Now by factoring out  $\text{Ann}M$ , we may assume  $M$  is faithful. For part (a'), let  $1 \leq i \leq s$  and  $\mathfrak{p}' \in \text{Spec}(R')$  with  $\text{ht} \mathfrak{p}' \leq i-1$ . Then  $\mathfrak{p}' = \mathfrak{p}R'$  for some  $\mathfrak{p} \in \text{Spec}(R)$  with  $\text{ht} \mathfrak{p} \leq i-1$ . If  $\mathfrak{p} \notin \text{Supp}(M/IM)$ , then  $IM'_{\mathfrak{p}'} = M'_{\mathfrak{p}'} = (x'_1, \dots, x'_i)M'_{\mathfrak{p}'}$ , where the last equality holds because  $x'_i R'_{\mathfrak{p}'} = R'_{\mathfrak{p}'}$ . Therefore we may assume  $\mathfrak{p} \in \text{Supp}(M/IM)$ . Since  $I$  satisfies condition  $G_s$  on  $M$  and  $\mathfrak{p} \in \text{Spec}(R)$  with  $\text{ht} \mathfrak{p} \leq i-1 < s$ , we have  $IM_{\mathfrak{p}} = (y_1, \dots, y_{i-1})M_{\mathfrak{p}}$  for some  $y_1, \dots, y_{i-1} \in I$ . By ( $\star$ ), this implies  $IM'_{\mathfrak{p}} = (x'_1, \dots, x'_{i-1})M'_{\mathfrak{p}}$ . Part (b') follows by employing the same argument.

Finally since for  $0 \leq i \leq s-1$ ,  $H'_i$  is a geometric  $i$ -residual intersection of  $IR'$  on  $M'$  and  $H'_s$  is a  $s$ -residual intersection of  $IR'$  on  $M'$ , we have  $\text{ht}((x'_1, \dots, x'_i)M' :_{R'} IM') \geq i$  for  $0 \leq i \leq s$  and  $\text{ht}((x'_1, \dots, x'_i)M' :_{R'} IM' + IM' :_{R'} M') \geq i+1$  for  $0 \leq i \leq s-1$ . Let  $k = R/\mathfrak{m}$ ,  $(\Lambda) = (\lambda_{ij}) \in R^{sn}$  and  $(\bar{\Lambda})$  the image of  $(\Lambda)$  in  $k^{sn}$ . Write  $\pi(\cdot)$  for the evaluation map sending  $z_{ij}$  to  $\lambda_{ij}$ . By [8, 3.1] for all  $i$ , there exists a dense open subset  $U$  of  $k^{sn}$  such that  $\text{ht}(\pi((x'_1, \dots, x'_i)M' :_{R'} IM')) \geq i$  and  $\text{ht}(\pi((x'_1, \dots, x'_i)M' :_{R'} IM' + IM' :_{R'} M')) \geq i+1$  whenever  $(\bar{\Lambda}) \in U$ . Since  $\pi((x'_1, \dots, x'_i)M' :_{R'}$



$IM')$ )  $\subseteq (x_1, \dots, x_i)M :_R IM$  and  $\pi(IM' :_{R'} M') \subseteq IM :_R M$ , then  $H_i$  is also a geometric  $i$ -residual intersection of  $I$  on  $M$  for  $0 \leq i \leq s-1$  and  $H_s$  is a  $s$ -residual intersection of  $I$  on  $M$ .  $\square$

Assume that  $M$  is Cohen-Macaulay. The ideal  $I$  is said to have the *Artin-Nagata property*  $AN_t^-$  on  $M$  if for every  $0 \leq i \leq t$  and every geometric  $i$ -residual intersection  $H$  of  $I$  on  $M$ , the module  $M/H$  is Cohen-Macaulay. In the next lemma we exhibit some basic facts about Artin-Nagata properties that will be useful in the proof of Theorem 3.8.

**Lemma 3.2.** *Let  $M$  be a Cohen-Macaulay module of dimension  $d$  over a Noetherian local ring  $R$  with infinite residue field. Let  $I$  be an  $R$ -ideal with  $\ell(I, M) = s$  satisfying  $G_s$  and  $AN_{s-1}^-$  on  $M$ . For general elements  $x_1, \dots, x_s$  of  $I$  on  $M$ , write  $H_i = (x_1, \dots, x_i)M :_M I$  for  $0 \leq i \leq s$ , then:*

- (a)  $x_{i+1}$  is regular on  $M/H_i$  and  $H_i = (x_1, \dots, x_i)M :_M x_{i+1}$  if  $0 \leq i \leq s-1$ .
- (b)  $M/H_i$  is unmixed of dimension  $d-i$ .
- (c)  $\text{depth } M/(x_1, \dots, x_i)M = d-i$ .
- (d)  $\text{depth } (M/(x_1, \dots, x_i)IM) \geq \min\{d-i, \text{depth } (M/IM)\}$ .
- (e)  $(x_1, \dots, x_i)M = H_i \cap IM$  if  $0 \leq i \leq s-1$ .
- (f) If  $\text{depth } (M/IM) \geq d-s+1$  then

$$(x_1, \dots, x_{s-1})M :_{I^2 M} I^\infty = (x_1, \dots, x_{s-1})M :_{I^2 M} I = (x_1, \dots, x_{s-1})M :_{I^2 M} x_s = (x_1, \dots, x_{s-1})IM.$$

- (g) Let  $\overline{M} = M/H_0$  then  $I$  satisfies  $G_s$  and  $AN_{s-1}^-$  on  $\overline{M}$ .

**Proof.** By Lemma 3.1, for general elements  $x_1, \dots, x_s$  of  $I$  on  $M$ , the module  $H_i$  is a geometric  $i$ -residual intersection of  $I$  on  $M$  for  $0 \leq i \leq s-1$  and  $H_s$  is a  $s$ -residual intersection of  $I$  on  $M$ . Thus parts (a), (b), (c), (e) and (g) follow as in the proofs of [26, 1.7] and [12, 2.3, 2.4]. To prove (d), we use induction on  $i$ . First when  $i=0$ , clearly  $\text{depth } M \geq \min\{d, \text{depth } (M/IM)\}$ . Assume  $\text{depth } (M/(x_1, \dots, x_i)IM) \geq \min\{d-i, \text{depth } (M/IM)\}$  for some  $0 \leq i < s$ . By (a) and (e),  $((x_1, \dots, x_i)M :_M x_{i+1}) \cap IM = (x_1, \dots, x_i)M$ . For  $i+1$ , we have

$$\begin{aligned} (x_1, \dots, x_i)IM \cap x_{i+1}IM &= x_{i+1}[(x_1, \dots, x_i)IM :_M x_{i+1}] \cap IM \\ &\subseteq x_{i+1}[(x_1, \dots, x_i)M :_M x_{i+1}] \cap IM \\ &= x_{i+1}(x_1, \dots, x_i)M \subseteq (x_1, \dots, x_i)IM \cap x_{i+1}IM. \end{aligned}$$

Thus we obtain an exact sequence:

$$0 \rightarrow x_{i+1}(x_1, \dots, x_i)M \rightarrow (x_1, \dots, x_i)IM \oplus x_{i+1}IM \rightarrow (x_1, \dots, x_{i+1})IM \rightarrow 0.$$

The element  $x_{i+1}$  is regular on  $IM$  and therefore on  $(x_1, \dots, x_i)M$  because  $(0_M :_M x_{i+1}) \cap (x_1, \dots, x_i)M \subseteq (0_M :_M x_{i+1}) \cap IM = 0_M$ , thus  $x_{i+1}(x_1, \dots, x_i)M \cong (x_1, \dots, x_i)M$  and  $x_{i+1}IM \cong IM$ . Therefore  $\text{depth } (x_{i+1}(x_1, \dots, x_i)M) \geq \min\{d, d-i+1\}$  and  $\text{depth } (x_{i+1}IM) \geq \min\{d, \text{depth } (M/IM) + 1\}$ . By induction hypothesis  $\text{depth } ((x_1, \dots, x_i)IM) \geq \min\{d-i+1, \text{depth } (M/IM) + 1\}$ , thus the above exact sequence yields  $\text{depth } ((x_1, \dots, x_{i+1})IM) \geq \min\{d-i, \text{depth } (M/IM) + 1\}$ . We finally conclude that  $\text{depth } (M/(x_1, \dots, x_{i+1})IM) \geq \min\{d-i-1, \text{depth } (M/IM)\}$ .

To show assertion (f), since

$$(x_1, \dots, x_{s-1})IM \subseteq (x_1, \dots, x_{s-1})M :_{I^2M} I \subseteq (x_1, \dots, x_{s-1})M :_{I^2M} I^\infty,$$

it is enough to check the equality locally at every prime ideal  $\mathfrak{p} \in \text{Ass}(M/(x_1, \dots, x_{s-1})IM)$ . By (d),  $\text{depth}(M/(x_1, \dots, x_{s-1})IM) \geq d - s + 1$ . Thus for every  $\mathfrak{p} \in \text{Ass}(M/(x_1, \dots, x_{s-1})IM)$ ,  $\text{ht } \mathfrak{p} \leq s - 1$  and hence either  $IM_{\mathfrak{p}} = M_{\mathfrak{p}}$  or  $IM_{\mathfrak{p}} = (x_1, \dots, x_{s-1})M_{\mathfrak{p}}$  since  $H_{s-1}$  is a geometric  $s - 1$ -residual intersection of  $I$  on  $M$ . Therefore, if  $IM_{\mathfrak{p}} = (x_1, \dots, x_{s-1})M_{\mathfrak{p}}$  then

$$[(x_1, \dots, x_{s-1})M :_{I^2M} I^\infty]_{\mathfrak{p}} = [(x_1, \dots, x_{s-1})M :_{I^2M} I]_{\mathfrak{p}} = (x_1, \dots, x_{s-1})IM_{\mathfrak{p}} = I^2M_{\mathfrak{p}}.$$

Otherwise by (e), we immediately obtain  $(x_1, \dots, x_{s-1})M_{\mathfrak{p}} = (H_{s-1})_{\mathfrak{p}}$ , and the latter coincides with the module  $[(x_1, \dots, x_{s-1})M :_{I^2M} I]_{\mathfrak{p}} = [(x_1, \dots, x_{s-1})M :_{I^2M} I^\infty]_{\mathfrak{p}}$ .

The full assertion now follows from part (a). □

The following lemma shows that the presence of  $G_d$  along with the Artin-Nagata condition  $AN_{d-2}^-$  is actually sufficient to obtain  $AN_d^-$ .

**Lemma 3.3.** (see [26, 1.9]) *Let  $M$  be a Cohen-Macaulay module of dimension  $d$  over a Noetherian local ring  $R$  and  $I$  an  $R$ -ideal satisfying  $G_d$  and  $AN_{d-2}^-$  on  $M$ . Then  $I$  satisfies  $AN_d^-$  on  $M$ .*

**Proof.** The claim follows from Lemma 3.2 (b). □

Now we show that, as in the  $\mathfrak{m}$ -primary case, minimal multiplicity yields reduction number one.

**Theorem 3.4.** *Let  $M$  be a Cohen-Macaulay module of dimension  $d$  over a Noetherian local ring  $R$  and let  $I$  be an  $R$ -ideal with  $\ell(I, M) = d$ . Assume  $\text{depth}(M/IM) \geq \min\{\dim(M/IM), 1\}$  and  $I$  satisfies  $G_d$  and  $AN_{d-2}^-$  on  $M$ . If  $I$  has minimal  $j$ -multiplicity on  $M$  then  $r(I, M) = 1$ .*

**Proof.** By adjoining variables to  $R$  and localizing, we may assume that the residue field is infinite. If  $\dim M/IM = 0$  then the assertion follows from [20, Theorem 2.9]. Now assume  $\dim M/IM > 0$ . For general elements  $x_1, \dots, x_d$  in  $I$ , let  $\overline{M} = M/((x_1, \dots, x_{d-1})M :_M I^\infty)$ . By Proposition 2.1 and Definition 2.4 the  $j$ -multiplicity can be computed using  $x_1, \dots, x_d$  thus  $j(I, M) = \lambda(I\overline{M}/I^2\overline{M})$ . From Definition 2.4 (see also the proof of Corollary 2.5), we obtain  $I^2M = x_dIM + (x_1, \dots, x_{d-1})M :_{I^2M} I^\infty$ . By Lemma 3.2 (f)

$$(x_1, \dots, x_{d-1})M :_{I^2M} I^\infty = (x_1, \dots, x_{d-1})IM$$

thus we conclude at once that the reduction number of  $I$  on  $M$  with respect to  $(x_1, \dots, x_d)$  is one. Now [25, 2.2] or [11, 8.6.6] imply that  $r(I, M) = 1$ . □

Our main application is the case when  $M = R$ . We obtain that ideals with residual intersection properties and minimal  $j$ -multiplicity have Cohen-Macaulay associated graded rings.

**Corollary 3.5.** *Let  $R$  be a Cohen-Macaulay local ring of dimension  $d$ . Let  $I$  be an  $R$ -ideal with  $\ell(I) = d$ . Assume  $\text{depth}(R/I) \geq \min\{\dim R/I, 1\}$  and  $I$  satisfies  $G_d$  and  $AN_{d-2}^-$ . If  $I$  has minimal  $j$ -multiplicity then the associated graded ring  $\text{gr}_I(R)$  is Cohen-Macaulay.*

**Proof.** The assertion follows from Theorem 3.4 and [12, 3.1].  $\square$

Notice that the ‘residual intersection assumptions’:  $\ell(I) = d$ ,  $\text{depth}(R/I) \geq \min\{\dim R/I, 1\}$ ,  $I$  satisfies  $G_d$  and  $AN_{d-2}^-$  are all vacuous in the 0-dimensional case and the condition on the minimal  $j$ -multiplicity becomes the usual assumption of minimal multiplicity found in the original work of Sally. If the ambient ring is Gorenstein she was able to prove that the associated graded ring is Gorenstein as well. We recover this result in Corollary 3.6.

We remark that the Artin-Nagata property  $AN_{d-2}^-$  holds if  $d \leq g + 1$ , or if  $I$  satisfies  $G_d$  and the sliding depth conditions  $\text{depth} R/I^j \geq \dim R/I - j + 1$  for  $1 \leq j \leq d - \text{ht} I + 1$ , a linear weakening of the Cohen-Macaulay property for consecutive powers of  $I$  (see [12, 2.1]). The depth inequalities are satisfied by strongly Cohen-Macaulay ideals, i.e., ideals whose Koszul homology modules are Cohen-Macaulay (see [26, 2.10]). Examples of ideals satisfying the latter condition are quite common – Cohen-Macaulay almost complete intersections, Cohen-Macaulay ideals generated by  $2 + \text{ht} I$  elements, perfect ideals of codimension two, perfect Gorenstein ideals of codimension three, and, more generally, any ideal in the linkage class of a complete intersection, namely, licci ideals (see [10, 1.11]).

**Corollary 3.6.** *Let  $R$  be a Gorenstein local ring of dimension  $d$ . Let  $I$  be an  $R$ -ideal with  $\ell(I) = d$ . Assume  $\text{depth} R/I^j \geq \dim R/I - j + 1$  for  $1 \leq j \leq d - \text{ht} I + 1$  and  $I$  satisfies  $G_d$ . If  $I$  has minimal  $j$ -multiplicity then the associated graded ring  $\text{gr}_I(R)$  is Gorenstein.*

**Proof.** The assertion follows from Theorem 3.4 and [12, 5.3].  $\square$

**Corollary 3.7.** *Let  $R$  be a Cohen-Macaulay local ring of dimension  $d$ . Let  $I$  be a strongly Cohen-Macaulay  $R$ -ideal with  $\ell(I) = d$ . If  $I$  has  $G_d$  and minimal  $j$ -multiplicity then the associated graded ring  $\text{gr}_I(R)$  is Cohen-Macaulay. In addition if  $R$  is Gorenstein then  $\text{gr}_I(R)$  is Gorenstein.*

**Proof.** The first assertion follows from Theorem 3.4 and [12, 3.2], and the second from [26, 2.10] and Corollary 3.6.  $\square$

In the next theorem we demonstrate that  $r(I, M) \leq 1$  implies the Cohen-Macaulayness of the associated graded module  $\text{gr}_I(M)$ .

**Theorem 3.8.** *Let  $M$  be a Cohen-Macaulay module of dimension  $d$  over a Noetherian local ring  $R$  and let  $I$  be an  $R$ -ideal with  $\ell(I, M) = d$ . Assume that  $I$  satisfies  $G_d$  and  $AN_{d-2}^-$  on  $M$ . If  $r(I, M) \leq 1$  then the associated graded module  $\text{gr}_I(M)$  is Cohen-Macaulay.*

**Proof.** By adjoining variables to  $R$  and localizing, we may assume that the residue field is infinite. Set  $g = \text{grade}(I, M)$ . Let  $x_1, \dots, x_g$  be general elements in  $I$  and  $x_1^*, \dots, x_g^*$  their initial forms in  $\text{gr}_I(R)$ . First we show that  $x_1^*, \dots, x_g^*$  form a  $\text{gr}_I(M)$ -regular sequence. By [27, 2.6], we only need to show  $(x_1, \dots, x_g)M \cap I^j M = (x_1, \dots, x_g)I^{j-1}M$  for every  $j \geq 1$ .

We use induction on  $j$  to prove  $(x_1, \dots, x_i)M \cap I^j M = (x_1, \dots, x_i)I^{j-1}M$  for every  $j \geq 1$  and  $0 \leq i \leq d$ . This is clear if  $j = 1$ . So we assume  $j \geq 2$  and the equality holds for  $j - 1$ . Set  $J = (x_1, \dots, x_d)$ . Since the reduction number of  $I$  on  $M$  is one, thus by [25, 2.2] or [11, 8.6.6] we have  $I^2 M = JIM$ . Therefore  $JM \cap I^j M = JI^{j-1}M$ . Now we use descending induction on  $i$  and assume  $(x_1, \dots, x_{i+1})M \cap I^j M = (x_1, \dots, x_{i+1})I^{j-1}M$ . Then

$$\begin{aligned}
(x_1, \dots, x_i)M \cap I^j M &= (x_1, \dots, x_i)M \cap (x_1, \dots, x_{i+1})I^{j-1}M && \text{by induction on } i \\
&= (x_1, \dots, x_i)M \cap ((x_1, \dots, x_i)I^{j-1}M + x_{i+1}I^{j-1}M) \\
&= (x_1, \dots, x_i)I^{j-1}M + (x_1, \dots, x_i)M \cap x_{i+1}I^{j-1}M \\
&= (x_1, \dots, x_i)I^{j-1}M + x_{i+1}[(x_1, \dots, x_i)M :_M x_{i+1}] \cap I^{j-1}M \\
&= (x_1, \dots, x_i)I^{j-1}M + x_{i+1}[(x_1, \dots, x_i)M \cap I^{j-1}M] && \text{by Lemma 3.2 (a) and (e)} \\
&= (x_1, \dots, x_i)I^{j-1}M + x_{i+1}(x_1, \dots, x_i)I^{j-2}M && \text{by induction on } j \\
&\subseteq (x_1, \dots, x_i)I^{j-1}M.
\end{aligned}$$

Set  $\delta(I, M) = d - g$ . Now we prove that the associated graded module  $\text{gr}_I(M)$  is Cohen-Macaulay by induction on  $\delta$ . If  $\delta = 0$ , the assertion follows because  $x_1^*, \dots, x_g^*$  form a  $\text{gr}_I(M)$ -regular sequence. Thus we may assume  $\delta(I, M) \geq 1$  and the theorem holds for smaller values of  $\delta(I, M)$ . In particular,  $d \geq g + 1$ . Again since  $x_1^*, \dots, x_g^*$  form a  $\text{gr}_I(M)$ -regular sequence, we may factor out  $x_1, \dots, x_g$  to assume  $g = 0$ . Now  $d = \delta(I, M) \geq 1$ . Set  $H_0 = 0_M :_M I$  and  $\overline{M} = M/H_0$ . Then  $\overline{M}$  is Cohen-Macaulay since  $I$  satisfies  $AN_d^-$  on  $M$  (see Lemma 3.3). By Lemma 3.2 (e), (a) and (g) and Lemma 3.3,  $IM \cap H_0 = 0$ ,  $\text{grade}(I, \overline{M}) \geq 1$ ,  $I$  still satisfies  $G_d$  and  $AN_d^-$  on  $\overline{M}$ . Furthermore  $\dim M = \dim \overline{M} = d$  and  $IM \cap H_0 = 0$  implies  $\ell(I, \overline{M}) = \ell(I, M) = d$ . Again by  $IM \cap H_0 = 0$ , there is a graded exact sequence

$$(1) \quad 0 \rightarrow H_0 \rightarrow \text{gr}_I(M) \rightarrow \text{gr}_I(\overline{M}) \rightarrow 0.$$

Notice that  $r(I, \overline{M}) \leq 1$ . Since  $\delta(I, \overline{M}) = d - \text{grade}(I, \overline{M}) < d = \delta(I, M)$ , by induction hypothesis  $\text{depth}(\text{gr}_I(\overline{M})) \geq d$ . Observe  $\text{depth}(H_0) \geq d$  since  $\overline{M}$  is Cohen-Macaulay. The Cohen-Macaulyness of  $\text{gr}_I(M)$  follows at once by the short exact sequence (1).  $\square$

Now we are ready to prove the main theorem.

**Theorem 3.9.** *Let  $M$  be a Cohen-Macaulay module of dimension  $d$  over a Noetherian local ring  $R$  and let  $I$  be an  $R$ -ideal with  $\ell(I, M) = d$ . Assume  $\text{depth}(M/IM) \geq \min\{\dim(M/IM), 1\}$  and  $I$  satisfies  $G_d$  and  $AN_{d-2}^-$  on  $M$ . If  $I$  has minimal  $j$ -multiplicity on  $M$  then the associated graded module  $\text{gr}_I(M)$  is Cohen-Macaulay.*

**Proof.** By Theorem 3.4, the reduction number of  $I$  on  $M$  is one. Now the assertion follows from Theorem 3.8.  $\square$

#### 4. ALMOST MINIMAL $j$ -MULTIPLICITY

We start by giving the definition of almost minimal  $j$ -multiplicity, which is the analogue of almost minimal multiplicity [20].

**Definition 4.1.** Let  $M$  be a finite module of dimension  $d$  over a Noetherian local ring  $R$  and  $I$  an  $R$ -ideal with analytic spread  $\ell(I, M) = d$ . We say that  $I$  has *almost minimal  $j$ -multiplicity on  $M$*  if  $j(I, M) = \lambda(I\overline{M}/I^2\overline{M}) + 1$ , where  $\overline{M} = M/((x_1, \dots, x_{d-1})M :_M I^\infty)$  and  $x_1, \dots, x_{d-1}$  are general in  $I$ .

Notice that by Lemma 2.3 the definition of almost minimal  $j$ -multiplicity is independent on the general sequence chosen in  $I$ .

**Remark 4.2.** If  $I$  has almost minimal  $j$ -multiplicity on  $M$  then

$$\lambda(I^2M/[x_dIM + (x_1, \dots, x_{d-1})M :_{I^2M} I^\infty]) = 1.$$

for any general sequence  $x_1, \dots, x_d$  in  $I$ .

**Notation and Discussion 4.3.** Let  $M$  be a finite module over a Noetherian local ring  $R$  and  $I$  an  $R$ -ideal. For every  $j \geq 1$ , let  $\widetilde{I^jM} = \cup_{t \geq 1} (I^{j+t}M :_M I^t)$  be the Ratliff-Rush filtration of  $I$  on  $M$  (see [16, 15, 20]). If  $\text{depth}_I M > 0$ , by [20, Lemma 3.1], there exists an integer  $n_0$  such that  $\widetilde{I^jM} = I^jM$  for every  $j \geq n_0$ . In particular, for every reduction  $J$  of  $I$  on  $M$  and every  $j \geq n_0$ , we obtain  $\widetilde{I^{j+1}M} = J\widetilde{I^jM} + I^{j+1}M$ . Thus the module  $N := \oplus_{j \geq 0} (\widetilde{I^{j+1}M}/J\widetilde{I^jM} + I^{j+1}M)$  has finitely many non-zero components. Since each component is a finitely generated  $R$ -module,  $N$  itself is finitely generated as an  $R$ -module; we denote with  $q = \mu(N)$  its minimal number of generators.

The next result is the key step in relating the reduction number of an  $I$ -adic filtration to invariants of the Ratliff-Rush filtration of  $I$  on  $M$ . The idea originated in the work of Rossi and Valla (see [18, 17, 20]). For clarity of exposition we state here the version for modules and ideals that are not necessarily  $\mathfrak{m}$ -primary.

**Theorem 4.4.** Use the notation of 4.3 and assume  $\text{depth}_I M > 0$ . Let  $J$  be an ideal generated by  $d$  general elements in  $I$ . Then

$$I^q \subseteq JI^{q-1} + (I^{q+j}M :_R \widetilde{I^jM})$$

for every positive integer  $j$ .

**Proof.** The proof is the same as the proof of [20, Theorem 4.1]. □

**Corollary 4.5.** Use the notation of 4.3 and assume  $\text{depth}_I M > 0$ . Let  $J$  be an ideal generated by  $d$  general elements in  $I$ . Then  $r(I, M) \leq t + q$ , where  $t = \min\{j \mid I^{j+1}M \subseteq J\widetilde{I^jM}\}$ .

**Proof.** The proof is the same as the proof of [20, Corollaries 4.1 and 4.2]. □

**Lemma 4.6.** *Let  $M$  be a Cohen-Macaulay module of dimension  $d$  over a Noetherian local ring  $R$ . Let  $I$  be an  $R$ -ideal with  $\ell(I, M) = d$ . Assume that  $I$  satisfies  $G_d$  and  $AN_{d-2}^-$  on  $M$  and let  $J$  be an ideal generated by  $d$  general elements in  $I$ . Then the lengths  $\lambda(I^j IM/JI^{j-1}IM)$ ,  $\lambda(\widetilde{I}^j IM/\widetilde{J}I^{j-1}IM)$  and  $\lambda(\widetilde{I}^j IM/I^j IM)$  are finite for all  $j \geq 1$ .*

**Proof.** Clearly  $\lambda(IM/JM) < \infty$  and  $\lambda(I^j IM/JI^{j-1}IM) < \infty$  for every  $j \geq 1$ , since  $IM$  and  $JM$  are the same on the punctured spectrum. To show the remaining assertions, observe that  $J$  has analytic spread  $d$  and satisfies  $G_d$  and  $AN_{d-2}^-$  on  $M$  as well, for instance by Lemma 3.1 and [26, 3.1]. Since  $r(J, M) = 0$ , by Theorem 3.8 the associated graded module  $\text{gr}_J(M)$  is Cohen-Macaulay. In particular,  $J$  is Ratliff-Rush closed on  $M$ , i.e.,  $\widetilde{J}^j M = J^j M$  for all  $j \geq 1$ . Thus on the punctured spectrum  $I$  is Ratliff-Rush closed on  $M$  as well, in particular  $\lambda(\widetilde{I}^j M/J^j M) < \infty$  for every  $j \geq 1$ .  $\square$

**Theorem 4.7.** *Let  $M$  be a Cohen-Macaulay module of dimension 2 over a Noetherian local ring  $R$  with infinite residue field. Let  $I$  be an  $R$ -ideal with  $\ell(I, M) = 2$ . Assume  $\text{depth}(M/IM) \geq \min\{\dim(M/IM), 1\}$  and  $I$  satisfies  $G_2$  and  $AN_0^-$  on  $M$ . Let  $x_1$  be a general element in  $I$ . If  $I$  has almost minimal  $j$ -multiplicity on  $M$  then*

- (a)  $x_1^*$  is regular on  $\text{gr}_I(M)_+$  ;
- (b)  $\text{depth}(\text{gr}_I(M)) \geq 1$ .

**Proof.** If  $\dim M/IM = 0$  then both claims follow from [20, Theorem 4.4]. Thus we may assume  $\text{depth}(M/IM) > 0$ .

Since  $I$  has almost minimal  $j$ -multiplicity on  $M$ , by Remark 4.2, for general elements  $x_1, x_2$  in  $I$ ,  $\lambda(I^2 M/[x_2 IM + x_1 M :_{I^2 M} I^\infty]) = 1$ . Set  $J = (x_1, x_2)$ . By Lemma 3.2 (f), we have  $x_1 M :_{I^2 M} I^\infty = x_1 IM$ , thus  $\lambda(I^2 M/JIM) = 1$ . Hence  $I^2 M = ab + JIM$  for some  $a \in I, b \in IM$  with  $ab \notin JIM$ . For  $j \geq 2$ , the multiplication by  $a$  gives a surjective map from  $I^j M/JI^{j-1}M$  to  $I^{j+1}M/JI^j M$ . Thus  $\lambda(I^j M/JI^{j-1}M) \leq 1$  for every  $j \geq 2$ .

Notice that  $x_1$  is regular on  $IM$  since  $(0 :_M x_1) \cap IM = 0$  (Lemma 3.2 (e)). Thus to prove that  $x_1^*$  is regular on  $\text{gr}_I(M)_+ = \text{gr}_I(IM)$  we only need to show  $x_1 IM \cap I^j IM = x_1 I^{j-1}IM$  for every  $j \geq 1$  by [27, 2.6] (see also [20, Lemma 1.1]). This is clear if  $j = 1$ ; hence we can assume  $j \geq 2$ . Let  $\overline{\phantom{x}}$  denote images in  $\overline{IM} = IM/x_1 M$  and set  $s = r_J(I, \overline{IM})$ . We claim that it is enough to show  $r_J(I, IM) = s$ . Indeed, if  $1 \leq j \leq s$  then  $JI^{j-1}IM + (x_1 M \cap I^j IM) = JI^{j-1}IM$ . This follows from the following easy inequality of lengths

$$\begin{aligned} 0 &< \lambda(I^j IM/JI^{j-1}IM + (x_1 M \cap I^j IM)) \\ &= \lambda(I^j IM/JI^{j-1}IM) - \lambda(JI^{j-1}IM + (x_1 M \cap I^j IM)/JI^{j-1}IM) \\ &= 1 - \lambda(JI^{j-1}IM + (x_1 M \cap I^j IM)/JI^{j-1}IM). \end{aligned}$$

On the other hand, if  $j \geq s+1 = r_J(I, IM) + 1$ , then  $I^j IM = JI^{j-1}IM$ . Thus we have for all  $j \geq 1$

$$(2) \quad x_1 IM \cap I^j IM = x_1 IM \cap JI^{j-1}IM.$$

Now we proceed by induction on  $j \geq 2$  as in the proof of Theorem 3.8:

$$\begin{aligned}
 x_1 IM \cap I^j IM &= x_1 IM \cap JI^{j-1} IM && \text{by (2)} \\
 &= x_1 IM \cap (x_1 I^{j-1} IM + x_2 I^{j-1} IM) \\
 &= x_1 I^{j-1} IM + (x_1 IM \cap x_2 I^{j-1} IM) \\
 &= x_1 I^{j-1} IM + x_2 [(x_1 IM :_{IM} x_2) \cap I^{j-1} IM] \\
 &= x_1 I^{j-1} IM + x_2 [x_1 IM \cap I^{j-1} IM].
 \end{aligned}$$

The last equality follows because  $x_1 IM \cap I^{j-1} IM \subseteq (x_1 IM :_{IM} x_2) \cap I^{j-1} IM \subseteq (x_1 M :_{I^2 M} x_2) \cap I^{j-1} IM = x_1 IM \cap I^{j-1} IM$  because  $j \geq 2$  and by Lemma 3.2 (f). Now using induction on  $j$ , we obtain

$$= x_1 I^{j-1} IM + x_2 [x_1 I^{j-2} IM] = x_1 I^{j-1} IM.$$

To complete the proof of (a), we still need to show that  $r_J(I, IM) = s$ . For this purpose we will use the Ratliff-Rush filtration  $\widetilde{I}^j IM$  as it is done for ideals of definition (see [20, Theorem 4.2]). As noticed earlier  $x_1$  is regular on  $IM$ . Thus, for instance by [20, Lemma 3.1], there exists an integer  $n_0$  such that  $I^j IM = \widetilde{I}^j IM$  for  $j \geq n_0$ , and

$$(3) \quad \widetilde{I^{j+1} IM} :_{IM} x_1 = \widetilde{I}^j IM \quad \text{for every } j \geq 0.$$

As before, let  $\overline{IM} = IM/x_1 M$  and  $\overline{\phantom{x}}$  denote images in  $\overline{IM}$ . There are two filtrations:

$$\overline{\mathbb{M}} : \overline{IM} \supseteq \overline{I^2 M} = I \overline{IM} \supseteq \dots \supseteq I^{j-1} \overline{IM} \supseteq \dots$$

and

$$\overline{\mathbb{N}} : \overline{IM} \supseteq \widetilde{IIM} = \widetilde{I} \overline{IM} \supseteq \dots \supseteq \widetilde{I^{j-1} IM} \supseteq \dots$$

Notice that  $\overline{\mathbb{M}}$  is an  $I$ -adic filtration and  $\overline{\mathbb{N}}$  is a good  $I$ -filtration on  $\overline{IM}$  (see [20, Page 9] for the definition of good filtration). Furthermore,  $I$  is an ideal of definition for  $\overline{IM}$ , i.e.,  $\lambda_R(\overline{IM}/I \overline{IM}) < \infty$ . Indeed,  $(x_1 M :_M x_2) \cap IM = x_1 M$  (see Lemma 3.2 (e)) which forces  $x_2 \in I$  to be regular on  $\overline{IM}$ , in turns this yields  $\lambda_R(\overline{IM}/I \overline{IM}) \leq \lambda_R(\overline{IM}/x_2 \overline{IM}) < \infty$ . Thus we are in the context of the filtrations treated in [20]. Since  $I^{j-1} \overline{IM} = \widetilde{I^{j-1} IM}$  for  $j \geq n_0$ , the associated graded modules  $\text{gr}_{\overline{\mathbb{M}}}(\overline{IM})$  and  $\text{gr}_{\overline{\mathbb{N}}}(\overline{IM})$  have the same Hilbert coefficients  $e_0$  and  $e_1$ . Again, because there exists an element in  $I$  which is regular on  $\overline{IM}$ , by [20, Lemmas 2.1 and 2.2] we have

$$(4) \quad s = \sum_{j \geq 0} \lambda(I^{j+1} \overline{IM}/x_2 I^j \overline{IM}) = e_1(\overline{\mathbb{M}}) = e_1(\overline{\mathbb{N}}) = \sum_{j \geq 0} \lambda(\widetilde{I^{j+1} IM}/x_2 \widetilde{I^j IM}).$$

Observe that the first equality holds because  $\lambda(I^{j+1} \overline{IM}/x_2 I^j \overline{IM}) = 1$  for all  $0 \leq j \leq s-1$  and  $\lambda(I^{j+1} \overline{IM}/x_2 I^j \overline{IM}) = 0$  for  $j \geq s = r_J(I, \overline{IM})$ .

We prove that  $\lambda(\widetilde{I^{j+1} IM}/x_2 \widetilde{I^j IM}) = \lambda(\widetilde{I^{j+1} IM}/\widetilde{JI^j IM})$  for every  $j \geq 0$ . Since

$$\widetilde{I^{j+1} IM}/x_2 \widetilde{I^j IM} \cong \widetilde{I^{j+1} IM}/(x_1 M \cap \widetilde{I^{j+1} IM} + x_2 \widetilde{I^j IM}),$$

we just need to show  $x_1 M \cap \widetilde{I^{j+1} IM} = x_1 \widetilde{I^j IM}$ . We first prove  $x_1 M \cap \widetilde{IIM} = x_1 IM$ . Since  $x_1 M \cap \widetilde{IIM} \supseteq x_1 IM$ , it suffices to show the equality locally at every associated prime ideal of  $M/x_1 IM$ .

By Lemma 3.2 (d), every  $\mathfrak{p} \in \text{Ass}(M/x_1IM)$  is not maximal. By the proof of Lemma 4.6,  $x_1M_{\mathfrak{p}} = \widetilde{IM}_{\mathfrak{p}} = IM_{\mathfrak{p}}$  thus  $x_1M_{\mathfrak{p}} \cap \widetilde{IM}_{\mathfrak{p}} = \widetilde{IM}_{\mathfrak{p}} = x_1IM_{\mathfrak{p}}$ . Therefore  $x_1M \cap \widetilde{IM} = x_1IM$ . Now for any  $j \geq 1$ ,  $x_1M \cap \widetilde{I^{j+1}IM} = x_1IM \cap \widetilde{I^{j+1}IM} = x_1(\widetilde{I^{j+1}IM} :_{IM} x_1) = x_1\widetilde{I^jIM}$ , where the last equality holds by (3).

Now (4) gives us

$$(5) \quad \sum_{j \geq 0} \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM}) = s,$$

and

$$p = \inf\{j \mid \widetilde{JI^jIM} = \widetilde{I^{j+1}IM}\} \leq s.$$

Let  $t = \inf\{j \mid I^{j+1}IM \subseteq \widetilde{JI^jIM}\}$ . Observe that  $t \leq p \leq s$  because  $I^{p+1}IM \subseteq \widetilde{I^{p+1}IM} = \widetilde{JI^pIM}$ . Let  $l$  be a positive integer such that for all  $0 \leq j \leq l$  we have  $I^{j+1}IM \cap JIM = JI^jIM$ . If  $t \leq l$ , then  $r_J(I, IM) \leq t \leq s$  and we are done. So we can assume that  $t > l$  and we have:

$$l < t \leq p \leq s.$$

By Corollary 4.5, the reduction number is bounded above by  $t + q$  where

$$(6) \quad q \leq \sum_{j \geq 0} \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM} + I^{j+1}IM).$$

To prove that  $r_J(I, IM) = s$ , it will be enough to show that  $\sum_{j \geq 0} \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM} + I^{j+1}IM) \leq s - t$ . Observe that for  $0 \leq j \leq l$ , we can relate  $\lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM} + I^{j+1}IM)$  to the difference of the length of the factors of the filtrations  $\mathbb{N}$  and  $\mathbb{M}$ . Indeed, for  $0 \leq j \leq l$  we have  $\widetilde{JI^jIM} \cap I^{j+1}IM = JI^jIM$  and therefore we obtain the following family of short exact sequences:

$$0 \longrightarrow \widetilde{JI^jIM} / JI^jIM \longrightarrow \widetilde{I^{j+1}IM} / I^{j+1}IM \longrightarrow \widetilde{I^{j+1}IM} / \widetilde{JI^jIM} + I^{j+1}IM \longrightarrow 0,$$

from which we obtain:

$$\begin{aligned} \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM} + I^{j+1}IM) &= \lambda(\widetilde{I^{j+1}IM} / I^{j+1}IM) - \lambda(\widetilde{JI^jIM} / JI^jIM) \\ &= \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM}) - \lambda(I^{j+1}IM / JI^jIM) \quad \text{for } 0 \leq j \leq l. \end{aligned}$$

For this we conclude that

$$(7) \quad \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM} + I^{j+1}IM) = \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM}) - 1 \quad 0 \leq j \leq l,$$

$$(8) \quad \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM} + I^{j+1}IM) \leq \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM}) - 1 \quad l+1 \leq j \leq t-1,$$

$$(9) \quad \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM} + I^{j+1}IM) = \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM}) \quad j \geq t.$$

Now by means of (5), (6), (7) and (8) we obtain

$$q \leq \sum_{j \geq 0} \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM} + I^{j+1}IM) \leq \sum_{j \geq 0} \lambda(\widetilde{I^{j+1}IM} / \widetilde{JI^jIM}) - t = s - t.$$

This concludes the proof of (a) since  $r_J(I, IM) \leq t + q \leq t + s - t = s$  (see Corollary 4.5).



Finally part (b) follows from (a). Indeed, by assumption  $\text{depth}(M/IM) > 0$  and by the exact sequence:

$$0 \rightarrow M/IM \rightarrow \text{gr}_I(M) \rightarrow \text{gr}_I(M)_+ \rightarrow 0,$$

we conclude  $\text{depth}(\text{gr}_I(M)) \geq \min\{\text{depth } M/IM, \text{depth}(\text{gr}_I(M)_+)\} \geq 1$ .

□

**Theorem 4.8.** *Let  $M$  be a Cohen-Macaulay module of dimension  $d$  over a Noetherian local ring  $R$ . Let  $I$  be an  $R$ -ideal with  $\ell(I, M) = d$ . Assume  $\text{depth}(M/IM) \geq \min\{\dim(M/IM), 1\}$  and  $I$  satisfies  $G_d$  and  $AN_{d-2}^-$  on  $M$ . If  $I$  has almost minimal  $j$ -multiplicity on  $M$  then  $\text{depth}(\text{gr}_I(M)) \geq d - 1$ .*

**Proof.** As in the proof of Theorem 3.4, we may assume that the residue field of  $R$  is infinite. We prove the theorem by induction on  $d$ . The case  $d = 2$  being proven in Theorem 4.7. Let  $d \geq 3$  and assume the theorem holds for  $d - 1$ . We first reduce to the case  $\text{grade}(I, M) \geq 1$ . If  $\text{grade}(I, M) = 0$ , let  $H_0 = 0 :_M I$ . As in the proof of Theorem 3.8, all assumptions still hold for the module  $M/H_0$ . Furthermore  $IM/H_0 = IM/(H_0 \cap IM) = IM$ ,  $\text{grade}(I, M/H_0) \geq 1$  and again as in the proof of Theorem 3.8,  $\text{depth}(\text{gr}_I(M)) \geq \text{depth}(\text{gr}_I(M/H_0))$ . So we are reduced to the case where the ideal  $I$  has at least one  $M$ -regular element. Thus if  $x_1$  is a general element in  $I$  then  $x_1$  is regular on  $M$ .

If  $\dim M/IM = 0$  then the assertion follows from [20, Theorem 4.4]. Thus we may assume  $\dim M/IM > 0$ . Let  $\overline{\phantom{x}}$  denote images in  $\overline{M} = M/x_1M$ . Observe that  $\overline{M}$  is a Cohen-Macaulay module of dimension  $d - 1$  and  $\ell(I, \overline{M}) = d - 1$ . Also  $I$  satisfies  $G_{d-1}$  and  $AN_{d-3}^-$  on  $\overline{M}$  by Lemma 3.2. Furthermore, observe  $\overline{M}/\overline{IM} \cong M/IM$  thus  $\text{depth}(\overline{M}/\overline{IM}) = \text{depth}(M/IM) \geq \min\{\dim M/IM, 1\} = \{\dim \overline{M}/\overline{IM}, 1\}$ . Clearly  $I$  has almost minimal  $j$ -multiplicity on  $\overline{M}$ . By induction hypothesis,  $\text{depth}(\text{gr}_I(\overline{M})) \geq d - 2$ .

Now we prove that  $x_1^*$  is regular on  $\text{gr}_I(M)$ . Since  $x_1$  is regular on  $M$ , by [27, 2.6] (see also [20, Lemma 1.1]), the claim follows if the intersections  $x_1M \cap I^jM = x_1I^{j-1}M$  hold for every  $j \geq 1$ . This is clear if  $j = 1$ . If  $j = 2$ , since  $x_1IM \subseteq x_1M \cap I^2M$ , it suffices to show the equality locally at every prime ideal  $\mathfrak{p} \in \text{Ass}(M/x_1IM)$ . By Lemma 3.2 (d),  $\text{depth}(M/x_1IM) \geq 1$ . Thus for every prime ideal  $\mathfrak{p} \in \text{Ass}(M/x_1IM)$ ,  $\mathfrak{p}$  is not the maximal ideal of  $R$  and hence either  $IM_{\mathfrak{p}} = M_{\mathfrak{p}}$  or  $IM_{\mathfrak{p}} = (x_1, \dots, x_{d-1})M_{\mathfrak{p}}$ . Therefore  $(x_1, \dots, x_{d-1})M_{\mathfrak{p}} \cap I^2M_{\mathfrak{p}} = (x_1, \dots, x_{d-1})IM_{\mathfrak{p}}$ . We use descending induction on  $i$  to prove  $(x_1, \dots, x_i)M_{\mathfrak{p}} \cap I^2M_{\mathfrak{p}} = (x_1, \dots, x_i)IM_{\mathfrak{p}}$  for every  $1 \leq i \leq d - 1$ . Assume  $(x_1, \dots, x_{i+1})M_{\mathfrak{p}} \cap I^2M_{\mathfrak{p}} = (x_1, \dots, x_{i+1})IM_{\mathfrak{p}}$ . Then

$$\begin{aligned} (x_1, \dots, x_i)M_{\mathfrak{p}} \cap I^2M_{\mathfrak{p}} &= (x_1, \dots, x_i)M_{\mathfrak{p}} \cap (x_1, \dots, x_{i+1})IM_{\mathfrak{p}} \\ &= (x_1, \dots, x_i)M_{\mathfrak{p}} \cap ((x_1, \dots, x_i)IM_{\mathfrak{p}} + x_{i+1}IM_{\mathfrak{p}}) \\ &= (x_1, \dots, x_i)IM_{\mathfrak{p}} + (x_1, \dots, x_i)M_{\mathfrak{p}} \cap x_{i+1}IM_{\mathfrak{p}} \\ &= (x_1, \dots, x_i)IM_{\mathfrak{p}} + x_{i+1}[(x_1, \dots, x_i)M_{\mathfrak{p}} :_{M_{\mathfrak{p}}} x_{i+1}] \cap IM_{\mathfrak{p}} \\ &= (x_1, \dots, x_i)IM_{\mathfrak{p}} + x_{i+1}(x_1, \dots, x_i)M_{\mathfrak{p}} \subseteq (x_1, \dots, x_i)IM_{\mathfrak{p}}. \end{aligned}$$

When  $j \geq 3$ , we have  $x_1 M \cap I^j M = x_1 M \cap I^2 M \cap I^j M = x_1 I M \cap I^{j-1} I M = x_1 I^{j-2} I M = x_1 I^{j-1} M$  since  $x_1^*$  is regular on  $\text{gr}_I(M)_+$  by Theorem 4.7. Finally since  $\text{depth}(\text{gr}_I(\bar{M})) \geq d - 2$  and  $x_1^*$  is regular on  $\text{gr}_I(M)$ , we have  $\text{depth}(\text{gr}_I(M)) \geq d - 1$ . □

Again our main application is the case when  $M = R$ . We obtain that ideals with residual intersection properties and almost minimal  $j$ -multiplicity have associated graded rings almost Cohen-Macaulay.

**Corollary 4.9.** *Let  $R$  be a Cohen-Macaulay local ring of dimension  $d$ . Let  $I$  be an  $R$ -ideal with  $\ell(I) = d$ . Assume  $\text{depth}(R/I) \geq \min\{\dim R/I, 1\}$  and  $I$  satisfies  $G_d$  and  $AN_{d-2}^-$ . If  $I$  has almost minimal  $j$ -multiplicity then  $\text{depth}(\text{gr}_I(R)) \geq d - 1$ .*

As noticed in Section 3 before Corollary 3.6, the assumptions on Corollary 4.9 are all vacuous in the 0-dimensional case and the condition on the almost minimal  $j$ -multiplicity becomes the usual assumption of almost minimal multiplicity found in Sally's conjecture and in the work of Rossi and Valla and Wang (see [23], [18], [28], [9], [3], [17], [4]). Thus Corollary 4.9 can be viewed as a positive answer to Sally's conjecture for arbitrary ideals.

Again the conclusion of Corollary 4.9 will hold true for  $I$  which is generically a complete intersection with  $\text{ht } I = d - 1$  or for strongly Cohen-Macaulay ideals satisfying  $G_d$ , in particular, for Cohen-Macaulay almost complete intersections, Cohen-Macaulay ideals generated by  $2 + \text{ht } I$  elements, perfect ideals of codimension two, perfect Gorenstein ideals of codimension three, and, more generally licci ideals.

We will finish our paper by the following examples.

**Example 4.10.** *Let  $S$  be a 3-dimensional Cohen-Macaulay local ring and  $x, y, z$  a system of parameters for  $S$ . We set  $R = S/(x^2 - yz)S$  and  $I = (x, y)R$ . Then  $I$  has minimal  $j$ -multiplicity. In particular, the associated graded ring  $\text{gr}_I(R)$  is Cohen-Macaulay, and if  $S$  is Gorenstein then  $\text{gr}_I(R)$  is Gorenstein as well.*

**Proof.** Observe that  $R$  is a Cohen-Macaulay local ring of dimension  $d = 2$ . By [13, 4.2],  $I$  is a Cohen-Macaulay ideal of height  $1 = d - 1$  which is generically a complete intersection and  $\ell(I) = 2$ . Let  $\xi$  be a general element of  $(x, y)S$  so that  $I = \xi R + yR$ . By [13, 4.2],

$$(10) \quad j(I) = \lambda_S(S/(x, y, z)S).$$

Let  $\bar{R} = R/(\xi R :_R (x, y)R^\infty) = R/(\xi R :_R y)$ , where the last equality holds since  $(x, y)R_{\mathfrak{p}}^\infty = yR_{\mathfrak{p}}$  for every associated prime ideal  $\mathfrak{p} \in \text{Ass}(R/\xi R)$ . We claim that

$$(11) \quad \lambda(\bar{I}\bar{R}/I^2\bar{R}) = \lambda_S(S/(x, y, z)S)$$

which together with (10) implies that  $I$  has minimal  $j$ -multiplicity. To prove (11) we may assume that  $\xi = x + \mu y$  for some  $\mu \in S$ . Notice that

$$\begin{aligned} \lambda(I\bar{R}/I^2\bar{R}) &= \lambda_S([(x, y)S + ((x^2 - yz, x + \mu y)S :_S y)] / [(x^2, xy, y^2)S + ((x^2 - yz, x + \mu y)S :_S y)]) \\ &= \lambda_S([(x, y)S + (x + \mu y, z + \mu x)S] / [(x, y)^2 S + (x + \mu y, z + \mu x)S]) \\ &= \lambda_S((x, y, z)S / [(x + \mu y, z + \mu x)S + (x, y, z)^2 S]) \\ &= \lambda_S(S / (x, y, z)S), \end{aligned}$$

where the second equality follows from  $(x + \mu y, z + \mu x)S = (x^2 - yz, x + \mu y)S :_S y$ . Indeed it is easy to see  $(x + \mu y, z + \mu x)S \subseteq (x^2 - yz, x + \mu y)S :_S y$ . For the other inclusion, let  $\rho \in (x^2 - yz, x + \mu y)S :_S y$  thus  $\rho y = \eta(x + \mu y) + \gamma(x^2 - yz)$  for some elements  $\eta$  and  $\gamma$  of  $S$ , which gives  $(\rho - \mu\eta + \gamma z)y = (\eta + \gamma x)x$ . As  $x, y$  form an  $S$ -regular sequence, it follows that  $\rho - \mu\eta + \gamma z = \alpha x$  and  $\eta + \gamma x = \alpha y$ . In particular,  $\rho = \alpha(x + \mu y) - \gamma(z + \mu x) \in (x + \mu y, z + \mu x)S$ . The last equality is verified by noticing that the  $S$ -module  $(x, y, z) / [(x + \mu y, z + \mu x) + (x, y, z)^2]$  is cyclic with annihilator  $(x, y, z)$ .

By Theorem 3.5 and Theorem 3.6, the associated graded ring  $\text{gr}_I(R)$  is Cohen-Macaulay and Gorenstein, respectively.  $\square$

**Example 4.11.** Let  $S = k[x, y, z]_{(x, y, z)}$  be a 3-dimensional regular local ring. Set  $R = S / (x^4 - y^2 z^2)S$  and  $I = (x^2, y^2)R$ . Computing  $j$ -multiplicity as in the above example one obtains  $j(I) = 8 = \lambda(I\bar{R}/I^2\bar{R})$  where  $\bar{R} = R / (\xi R : I)$  and  $\xi$  is a general element in  $I$ . Now again by Example 4.10 the associated graded ring  $\text{gr}_I(R)$  is Gorenstein. Indeed,  $\text{gr}_I(R) \cong \text{gr}_{(x^2, y^2)S}(S) / ((x^4 - y^2 z^2)^*)$ .

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